

# Quantum Critical Dynamics of A Qubit Coupled to An Isotropic Lipkin-Meshkov-Glick Bath

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We explore a dynamic signature of quantum phase transition (QPT) in an isotropic Lipkin-Meshkov-Glick (LMG) model by studying the time evolution of a central qubit coupled to it. We evaluate exactly the time-dependent purity, which can be used to measure quantum coherence, of the central qubit. It is found that distinctly different behaviors of the purity as a function of the parameter reveal clearly the QPT point in the system. It is also clarified that the present model is equivalent to an anti Jaynes-Cummings model under certain conditions.

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## I. INTRODUCTION

Quantum phase transitions (QPTs) [1] in spin systems, e.g., the XY model [2], the Lipkin-Meshkov-Glick (LMG) model [3], and the Dicke model [4], have aroused much interest in recent years. Most of these efforts have addressed possible connections of quantum entanglement measures, such as the concurrence, the entanglement entropy, and the negativity, with the QPTs in the systems. The scaling behavior [2, 5] of the entanglement demonstrate well the quantum criticality of these systems. On the other hand, being related to quantum measurement theory and quantum decoherence problems, theoretical studies of the bath influence on the dynamic property of a central system have also attracted much attention. Ref. [6] claimed to find “an intrinsic limit to quantum coherence due to spontaneous symmetry breaking” in Lieb-Mattis model; the relationship between entanglement dynamics and paramagnet-ferromagnet phase transition was explored in Ref. [7]; it was shown in Ref. [8] that the Loschmidt echo decay enhanced at the critical point can be a signature of QPT in the transverse Ising model. In fact, further deeper studies on the the relevant issues of open quantum systems not only provides us a better understanding of the quantum-classical crossover, but also promises important potential applications in quantum information processing [9].

In this paper, integrating coherently the above two interesting topics: quantum phase transition and quantum open system [10], we elaborate how the QPT of the “bath” influences the dynamics of a central qubit coupled to it. It is shown that when the bath is in different phases, the purity of the central qubit exhibit distinctly different behaviors in two different phases. Moreover, it is also illustrated that under certain conditions our model is equivalent to an anti Jaynes-Cummings (anti J-C) model [11].

The paper is organized as follows. In Sec. II, we introduce the LMG model and summarize the main properties of this model. In Sec. III, we study the dynamic evolution of a central qubit coupled to a bath described by

an isotropic LMG model, which is exactly solvable. In Sec. IV, We evaluate the purity of the central qubit in the symmetry broken phase and the symmetric phase, respectively. The QPT of the bath is well indicated by the behavior of the purity. In Sec. V, a connection between the current model and an anti J-C model is established. Section VI presents our summary and conclusion.

## II. LIPKIN-MESHKOV-GLICK (LMG) MODEL FOR QUANTUM PHASE TRANSITION

We consider a central qubit (two-level system) that couples to a multi-qubit bath, which is described by the LMG model [3]

$$H_B = -\frac{\lambda}{N} \sum_{i < j}^N (\sigma_i^x \sigma_j^x + \gamma \sigma_i^y \sigma_j^y) - \sum_{i=1}^N \sigma_i^z, \quad (1)$$

where  $\sigma_i^\alpha$ ,  $\alpha = x, y, z$  ( $i = 1, 2, \dots, N$ ) are the Pauli matrices of the  $i$ -th atom,  $\lambda/N$  denotes the coupling strength, which is inversely proportional to the atom number  $N$ . This Hamiltonian contains long-range interactions, i.e., every spin in the bath interacts with all the others. In the isotropic case,  $\gamma = 1$ , the Hamiltonian is diagonal in the Dicke representation

$$H_B = -\frac{2\lambda}{N} \left[ \mathbf{J}_N^2 - (J_N^z)^2 - \frac{N}{2} \right] - 2J_N^z, \quad (2)$$

and the ground state of  $H_B$  lies in the subspace spanned by the Dicke states  $\{|N/2, M\rangle, M = -N/2, \dots, N/2\}$  [12]. Here,  $\mathbf{s} = \vec{\sigma}/2$ ,  $J_N^\alpha = 1/2 \sum_{i=1}^N \sigma_i^\alpha$  and

$$\begin{aligned} \mathbf{J}_N^2 \left| \frac{N}{2}, M \right\rangle &= \frac{N}{2} \left( \frac{N}{2} + 1 \right) \left| \frac{N}{2}, M \right\rangle, \\ J_N^z \left| \frac{N}{2}, M \right\rangle &= M \left| \frac{N}{2}, M \right\rangle. \end{aligned} \quad (3)$$

The eigenenergy corresponding to  $|N/2, M\rangle$  is  $2\lambda M^2/N - 2M - \lambda N/2$ . Hence, the ground state  $|G\rangle$  is  $\lambda$ -dependent

[13], i.e.,

$$|G\rangle = \begin{cases} |\frac{N}{2}, \frac{N}{2}\rangle, & (0 < \lambda < 1), \\ |\frac{N}{2}, I(\lambda)\rangle, & (\lambda > 1) \end{cases} \quad (4)$$

where  $I(\lambda)$  is the integer nearest to  $N/2\lambda$ . The level crossing at  $\lambda = 1$  leads to the occurrence of a QPT. This point is also a symmetry breaking point [13, 14]: when  $0 < \lambda < 1$ , the ground state of the bath is unique and fully polarized in the magnetic field direction, and thus the bath is in a symmetry broken phase; when  $\lambda > 1$ , the ground state is infinitely degenerate and thus the bath is in a symmetric phase. We below elaborate how the dynamic evolution of the purity [15] (a measure of quantum coherence) depends on the coupling strength between the central spin and the bath; in particular, we observe that the purity shows distinctly different behaviors in the two phases, which may be used to reveal the QPT point in the bath.

### III. DYNAMICS OF A CENTRAL QUBIT COUPLED TO AN ISOTROPIC LMG MODEL

A spin-bath model is described by the total Hamiltonian  $H = H_B + H_S + H_{SB}$  [16, 17], where  $H_S = -2s_z$  is the free Hamiltonian of the central qubit  $S$ .  $H_{SB}$  denotes the coupling between  $S$  and the bath  $B$ . Specifically, the total Hamiltonian can be written as

$$H = -\frac{\lambda}{N} \sum_{i < j}^N (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) - \sum_{i=1}^N \sigma_i^z \quad (5)$$

$$+ \lambda' \sum_i^N (\sigma_i^x \sigma^x + \sigma_i^y \sigma^y) - \sigma^z,$$

where  $\sigma^\alpha$ ,  $\alpha = x, y, z$ , are the Pauli operators of the central qubit;  $\lambda'$  is the coupling strength between the central qubit and the bath. In the Dicke representation, the above Hamiltonian can be rewritten as

$$H = -\frac{\lambda}{N} [J_N^+ J_N^- + J_N^- J_N^+ - N] - 2J_N^z \quad (6)$$

$$- 2\lambda' (s_+ J_N^- + s_- J_N^+) - (2s_z),$$

where  $J_N^\pm = J_N^x \pm iJ_N^y$  and  $s_\pm = s^x \pm is^y$  are the ladder operators of the  $N$ -qubit bath and the central qubit, respectively. For simplicity, we denote the two eigenstate of the central qubit as  $|\uparrow\rangle = |1/2, 1/2\rangle$  and  $|\downarrow\rangle = |1/2, -1/2\rangle$ , and  $s_z |\uparrow\rangle = |\uparrow\rangle/2$ ,  $s_z |\downarrow\rangle = -|\downarrow\rangle/2$ . In an invariant subspace  $\mathcal{H}_M$  of  $H$  spanned by the ordered basis vector  $\{|N/2, M\rangle \otimes |\uparrow\rangle, |N/2, M+1\rangle \otimes |\downarrow\rangle\}$ , the total Hamiltonian can be expressed as a quasideagonal matrix with the diagonal blocks

$$H_M = \begin{bmatrix} \alpha & \zeta \\ \zeta & \beta \end{bmatrix}, \quad (7)$$

where

$$\alpha = -\frac{\lambda}{2N} [N^2 - 4M^2] - 2M - 1, \quad (8)$$

$$\beta = -\frac{\lambda}{2N} [N^2 - 4(M+1)^2] - 2(M+1) + 1,$$

$$\zeta = -\lambda' \sqrt{N(N+2) - 4M(M+1)}.$$

A straightforward calculation determines the two eigenvalues  $x_1$  and  $x_2$  of  $H_M$  as

$$x_1 = \frac{1}{2}[(\alpha + \beta) + \sqrt{(\alpha - \beta)^2 + 4\zeta^2}], \quad (9)$$

$$x_2 = \frac{1}{2}[(\alpha + \beta) - \sqrt{(\alpha - \beta)^2 + 4\zeta^2}],$$

and the eigenstate  $|\Psi_1\rangle$  corresponding to  $x_1$  is

$$|\Psi_1\rangle = a \left| \frac{N}{2}, M \right\rangle \otimes |\uparrow\rangle + b \left| \frac{N}{2}, M+1 \right\rangle \otimes |\downarrow\rangle,$$

where

$$a = \frac{\zeta}{\sqrt{(\alpha - x_1)^2 + \zeta^2}}, \quad (10)$$

$$b = \frac{x_1 - \alpha}{\sqrt{(\alpha - x_1)^2 + \zeta^2}}.$$

We would like to point out that, in the symmetric phase, all  $x_1$ ,  $x_2$ ,  $a$ , and  $b$  are functions of  $I(\lambda)$ , i.e.,  $F = F[I(\lambda)]$  with  $F = x_1, x_2, a$ , and  $b$ . The dynamic evolution operator  $U(t) = \exp[-iHt]$  in the subspace  $\mathcal{H}_M$  can be expressed in terms of  $x_1$ ,  $x_2$ ,  $a$ , and  $b$

$$U_M(t) = \begin{bmatrix} a^2 e^{-ix_1 t} + b^2 e^{-ix_2 t} & ab(e^{-ix_1 t} - e^{-ix_2 t}) \\ ab(e^{-ix_1 t} - e^{-ix_2 t}) & b^2 e^{-ix_1 t} + a^2 e^{-ix_2 t} \end{bmatrix}, \quad (11)$$

We wish to mention that the above dynamic evolution (11) is valid only for the cases when  $-N/2 \leq M < N/2$ , because  $\{|N/2, M\rangle \otimes |\uparrow\rangle, |N/2, M+1\rangle \otimes |\downarrow\rangle\}$  is a two-dimensional invariant subspace for these cases. But for the case  $M = N/2$ ,  $\{|N/2, M\rangle \otimes |\uparrow\rangle\}$  is a one-dimensional invariant subspace, i.e.,  $|N/2, N/2\rangle \otimes |\uparrow\rangle$  is an eigenstate of the total Hamiltonian (6), and its corresponding eigenenergy is  $-(N+1)$ . Thus the dynamic evolution of this state  $|N/2, N/2\rangle \otimes |\uparrow\rangle$  is different from Eq. (11). We will discuss this point in the next section.

### IV. PURITY OF THE CENTRAL QUBIT AS A WITNESS OF QUANTUM PHASE TRANSITION

Based on the above results, we now solve the Schrodinger equation that describes the dynamics of the purity of the central qubit. To highlight the influence of the QPT of the bath on the coupled central qubit, it is assumed that the bath and the central qubit are initially in the ground state  $|G\rangle$  (4) and a pure superposition state

$c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle$ , respectively. The evolution of the total system (the bath plus the central qubit) is

$$|\Psi_{N+1}(t)\rangle = e^{-iHt} |G\rangle \otimes (c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle), \quad (12)$$

and the reduced density matrix of the central qubit is

$$\rho^S(t) = \text{Tr}_B |\Psi_{N+1}(t)\rangle \langle \Psi_{N+1}(t)|, \quad (13)$$

where  $\text{Tr}_B$  means tracing out the degree of freedom of the bath.

The purity  $P$  of the central qubit is defined as

$$\begin{aligned} P &= \text{Tr}_S \{[\rho^S(t)]^2\} \\ &= [\rho_{\uparrow\uparrow}^S(t)]^2 + [\rho_{\downarrow\downarrow}^S(t)]^2 + 2 |\rho_{\uparrow\downarrow}^S(t)|^2, \end{aligned} \quad (14)$$

which can be used to measure the quantum coherence. For a pure state, the purity equals to unity, while for a mixed state the purity is less than unity. The decay of purity indicates the loss of quantum coherence [15].

### A. Purity of the central qubit in two phases

#### 1. Symmetric phase

When  $\lambda > 1$ , the bath is in the symmetric phase and  $I(\lambda) < N/2$ . As mentioned above, we can apply the evolution matrix (11) to obtain the reduced density matrix  $\rho^S(t)$  of the central qubit with the matrix elements defined by

$$\begin{aligned} \rho_{\uparrow\uparrow}^S(t) &= \sqrt{[|c_\uparrow|^2 f(\lambda, t) + |c_\downarrow|^2 h(\lambda, t)]^2}, \\ \rho_{\downarrow\downarrow}^S(t) &= \sqrt{[|c_\downarrow|^2 g(\lambda, t) + |c_\uparrow|^2 i(\lambda, t)]^2}, \\ |\rho_{\uparrow\downarrow}^S(t)| &= \sqrt{|c_\uparrow c_\downarrow^*|^2} g(\lambda, t) \times f(\lambda, t) = |\rho_{\downarrow\uparrow}^S(t)|, \end{aligned} \quad (15)$$

where

$$\begin{aligned} f(\lambda, t) &= a^4 + b^4 + 2a^2b^2 \cos[(x_1 - x_2)t], \\ g(\lambda, t) &= (a')^4 + (b')^4 + 2(a')^2(b')^2 \cos[(x'_1 - x'_2)t], \\ h(\lambda, t) &= 2(a')^2(b')^2 \{1 - \cos[(x'_1 - x'_2)t]\}, \\ i(\lambda, t) &= 2a^2b^2 \{1 - \cos[(x_1 - x_2)t]\}, \end{aligned} \quad (16)$$

and the parameters  $x'_1$ ,  $x'_2$ ,  $a'$ , and  $b'$  are defined by  $F' = F[I(\lambda) - 1]$  with  $F' = x'_1$ ,  $x'_2$ ,  $a'$ , and  $b'$ . This subtle change from  $(x_1, x_2, a, b)$  to  $(x'_1, x'_2, a', b')$  is due to the fact that  $|G\rangle \otimes |\uparrow\rangle$  and  $|G\rangle \otimes |\downarrow\rangle$  belong to two different invariant subspace  $\mathcal{H}_M$  and  $\mathcal{H}_{M-1}$ . If, for simplicity, we assume that the central qubit is initially in the superposition state  $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ , we obtain from Eq. (14) the exact expression of the purity of the central qubit

$$\begin{aligned} P &= \frac{1}{4}[f(\lambda, t) + h(\lambda, t)]^2 + \frac{1}{4}[i(\lambda, t) + g(\lambda, t)]^2 \\ &\quad + \frac{1}{2}g(\lambda, t)f(\lambda, t). \end{aligned} \quad (17)$$

#### 2. Symmetry broken phase

When  $0 < \lambda < 1$ , the bath is in the symmetry broken phase and the ground state is the fully polarized state  $|G\rangle = |N/2, N/2\rangle$ .  $|G\rangle \otimes |\uparrow\rangle$  is an eigenstate of the total Hamiltonian (5), and its corresponding eigenenergy is  $-(N+1)$ . Thus the dynamic evolution of this state is  $\exp[i(N+1)t]$ . While the dynamic evolution of the other state  $|G\rangle \otimes |\downarrow\rangle$  can be obtained following the way mentioned above Eq.(11) with  $M = N/2 - 1$ . After a similar procedure, if the central qubit is initially prepared in the superposition state  $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ , the exact expression of the purity in the symmetry broken phase can be written as

$$P = \frac{1}{4}[1 + \tilde{h}(\lambda, t)]^2 + \frac{1}{4}\tilde{g}^2(\lambda, t) + \frac{1}{2}\tilde{g}(\lambda, t), \quad (18)$$

where

$$\begin{aligned} \tilde{g}(\lambda, t) &= \tilde{a}^4 + \tilde{b}^4 + 2\tilde{a}^2\tilde{b}^2 \cos[(\tilde{x}_1 - \tilde{x}_2)t], \\ \tilde{h}(\lambda, t) &= 2\tilde{a}^2\tilde{b}^2 \{1 - \cos[(\tilde{x}_1 - \tilde{x}_2)t]\}. \end{aligned}$$

The new parameters  $\tilde{x}_1$ ,  $\tilde{x}_2$ ,  $\tilde{a}$ , and  $\tilde{b}$  are given by  $\tilde{F} = F[N/2 - 1]$  with  $\tilde{F} = \tilde{x}_1$ ,  $\tilde{x}_2$ ,  $\tilde{a}$ , and  $\tilde{b}$ .

To obtain the exact result of the purity of the central qubit, we need to know the coupling strength  $\lambda'$  [18] between the central qubit and the bath. In different references, this coupling strength is treated differently. For example, this coupling strength was assumed to be inversely proportional to the qubit number of the bath in Ref. [6], while it was assumed to be inversely proportional to the square root of the spin number in some other references [7, 16]. We hereafter denote the two cases with the above two different coupling strengths as Cases I and II, respectively. Generally speaking, when the central qubit is identical to the qubits of the bath, the coupling strength between the central qubit and the bath should be equal to the coupling strength between the qubits of the bath. This is Case I, and we will elaborate it in next subsection. Besides Case I, Case II will also be discussed later. The calculation for Case II is the same as that for Case I except that  $\lambda'$  in Case I is changed to  $\sqrt{N}\lambda'$ . The two different coupling strengths will lead to different behaviors of the purity.

### B. The coupling strength inversely proportional to the spin number of the bath (Case I)

Firstly let us consider Case I. Similar to the coupling mechanism in Ref. [6], we assume that the coupling strength  $\lambda'$  is just the coupling strength  $\lambda/N$  between any two qubits of the bath. Since the coupling strength  $\lambda/N$  between qubits in the LMG model is inversely proportional to the spin number  $N$ , the system is extensive.

Fig. 1 clearly shows that, in the symmetry breaking phase, the purity of the central qubit remains as a con-

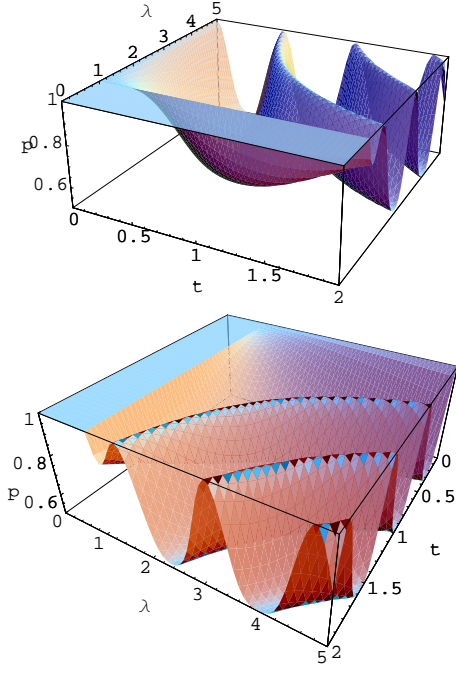


FIG. 1: (Color online) Two different view angles on the dynamic evolution of the purity  $P$  as functions of  $\lambda$  and  $t$  (Case I). The QPT (symmetry breaking) at  $\lambda = 1$  is well signatured. The purity saturates when  $N$  becomes large. Here we have chosen the qubit number of the environment  $N = 5000$ .

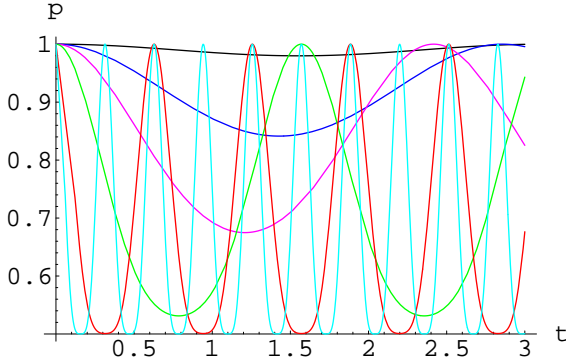


FIG. 2: (Color online) Dynamic evolution of the purity  $P$  as a function of time  $t$  for different  $\lambda$ . In the symmetry broken phase ( $0 < \lambda < 1$ ),  $P$  remains as a constant unity. In the symmetric phase ( $\lambda > 1$ ),  $P$  ranges from 0.5 to 1, and the period of the oscillation decreases as  $\lambda$  increases. The curves with different colors represent  $\lambda = 1.01$ ,  $\lambda = 1.1$ ,  $\lambda = 1.3$ ,  $\lambda = 2$ , and  $\lambda = 5$  ( $N = 5000$ ).

stant unity, i.e., the qubit preserves its quantum coherence all the time. The decay of purity becomes vanishingly small in the thermodynamic limit. While in the symmetric phase, the purity varies periodically, as shown in Fig. 2.

The constant purity  $P = 1$  of the central qubit for the

symmetry broken phase of the bath can also be verified through another approach. The ground state of the LMG model in the symmetry broken phase ( $0 < \lambda < 1$ ) is  $|G\rangle = |N/2, N/2\rangle$ . The direct product of the bath  $|G\rangle$  and the central qubit  $c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle$  can be expanded in the angular momentum coupling representation

$$|G\rangle \otimes (c_\uparrow |\uparrow\rangle + c_\downarrow |\downarrow\rangle) \approx c_\uparrow \left| \frac{1}{2}(N+1), \frac{1}{2}(N+1) \right\rangle + c_\downarrow \left| \frac{1}{2}(N-1), \frac{1}{2}(N-1) \right\rangle, \quad (19)$$

where we have used the Clebsch-Gordan (C-G) coefficient

$$\begin{aligned} \left| \frac{N}{2}, \frac{N}{2} \right\rangle \otimes |\uparrow\rangle &= \left| \frac{1}{2}(N+1), \frac{1}{2}(N+1) \right\rangle \\ \left| \frac{N}{2}, \frac{N}{2} \right\rangle \otimes |\downarrow\rangle &= \frac{\sqrt{N}}{\sqrt{N+1}} \left| \frac{1}{2}(N-1), \frac{1}{2}(N-1) \right\rangle \\ &\quad + \frac{1}{\sqrt{N+1}} \left| \frac{1}{2}(N+1), \frac{1}{2}(N-1) \right\rangle. \end{aligned} \quad (20)$$

The total Hamiltonian (5) of the qubit and the bath can be rewritten as

$$H = -\frac{\lambda}{N} [2\mathbf{J}_{N+1}^2 - 2(J_{N+1}^z)^2 - (N+1)] - 2J_{N+1}^z. \quad (21)$$

Through the above approximation (19), both  $|N/2, N/2\rangle \otimes |\uparrow\rangle$  and  $|N/2, N/2\rangle \otimes |\downarrow\rangle$  are the eigenstate of the total Hamiltonian with the eigenenergy  $-(N+1)$  and  $2\lambda/N - (N-1)$  respectively, and then the dynamic evolution of the two states are obvious. After a straightforward derivation, the reduced density matrix of the qubit is expressed as

$$\begin{aligned} \rho^S(t) &= |c_\uparrow|^2 |\uparrow\rangle \langle \uparrow| + |c_\downarrow|^2 |\downarrow\rangle \langle \downarrow| \\ &\quad + c_\uparrow c_\downarrow^* e^{i(\frac{2\lambda}{N} + 2)t} |\uparrow\rangle \langle \downarrow| + h.c., \end{aligned} \quad (22)$$

and the purity of  $\rho^S(t)$  remains as unity by applying Eq. (14). It is thus proven that the qubit preserves its quantum coherence when the bath is in its symmetry broken phase ( $0 < \lambda < 1$ ).

### C. The coupling strength inversely proportional to the square root of the spin number of the bath (Case II)

We now turn to Case II [7, 16]. Being different from Case I, the purity of the central qubit does not preserve its coherence when the bath is in the symmetry broken phase (see Fig.3), although the purity varies also periodically in the symmetric phase (Fig.4). However, both the range and the pattern of its time-dependance in the two phases are different from those in Case I. We also remark that the purity  $P$  saturates in the symmetry broken phase when  $N$  increases, just like that in the symmetric phase of Case I; while the dynamic behavior of the purity in the

symmetric phase depends on  $N$  with the period being inversely proportional to  $\sqrt{N}$  approximately, as analyzed later.

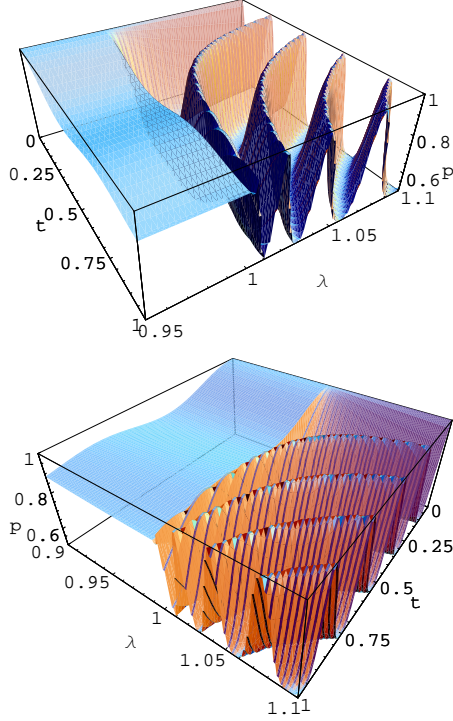


FIG. 3: (Color online) Dynamic evolution of the purity  $P$  as functions of  $\lambda$  and  $t$  (Case II) in the symmetric phase and symmetry broken phase. Clearly the purity varies in distinctly different manners in the two phases, which may be considered as an indication of QPT at the critical point  $\lambda = 1$ . The purity reaches a steady state in the symmetry broken phase, while the pattern of the purity will always change with  $N$  in the symmetric phase. Here we choose  $N = 1000$ .

## V. EQUIVALENCE TO AN ANTI JAYNES-CUMMINGS MODEL

In this section we show the equivalence between the above model and an anti J-C model with an intensity-dependent coupling strength [19]. With this observation, we can exactly solve the dynamical equation about time evolution.

### A. Symmetry broken phase

When the bath is in the symmetry broken phase, our model may be recast into an anti J-C model. Actually the equivalence between the LMG model and Dike model was just studied recently [20].

In the symmetry broken phase ( $0 < \lambda < 1$ ), the ground state  $|N/2, N/2\rangle$  of the bath corresponds to a low exci-

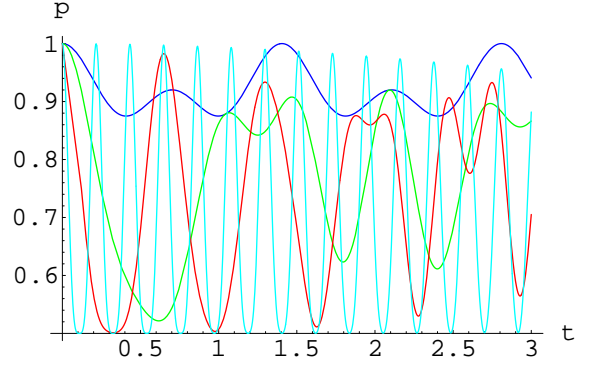


FIG. 4: (Color online) Dynamic evolution of the purity  $P$  as a function of time  $t$  for different  $\lambda$ . In the symmetry breaking phase ( $0 < \lambda < 1$ ), the dynamic behavior of  $P$  saturates when  $N$  increases, which ranges from about 0.88 to 1 with the period about 1.4. In the symmetric phase ( $\lambda > 1$ ),  $P$  ranges from 0.5 to 1, and the period of the oscillation decreases as  $\lambda$  increases. The curves of different colors represent  $\lambda = 1.0001$ ,  $\lambda = 1.0003$ ,  $\lambda = 1.002$ ,  $\lambda = 1.02$  ( $N = 1000$ ).

tation Fock state  $|0\rangle$  after the H-P transformation. The mean photon number  $n = \langle d^\dagger d \rangle = 0$ . Hence we can directly expand the Holstein-Primakoff (H-P) transformation [21] to the first-order [22]

$$\begin{aligned} J_N^+ &= \sqrt{N}d, J_N^- = (J_N^+)^\dagger \\ J_N^z &= N/2 - d^\dagger d, \end{aligned}$$

and the Hamiltonian (6) can be rewritten as

$$H = 2(1 - \lambda)d^\dagger d - N - 2\lambda'\sqrt{N}(s_+d^\dagger + s_-d) - 2s_z. \quad (23)$$

Let us recall that the anti J-C Hamiltonian can be rewritten as

$$H_{AJC} = \nu d^\dagger d - k(\sigma_+d^\dagger + \sigma_-d) + \frac{1}{2}\omega\sigma_z, \quad (24)$$

where  $d^\dagger$  and  $d$  are the creation and annihilation operators of the single-mode quantized field with the frequency  $\nu$ ;  $-k$  is the coupling strength between the field and the two-level system;  $\omega$  is the level spacing between the two-level system;  $\sigma_+ = (\sigma_x + i\sigma_y)/2$  and  $\sigma_- = (\sigma_x - i\sigma_y)/2$ . Hence, the model described by Eq. (23) is an anti J-C model (24) with  $\nu = 2(1 - \lambda)$ ,  $\omega = -2$ , and  $k = 2\lambda'\sqrt{N}$ . We now illustrate that the boson mode characterized by  $d$  and  $d^\dagger$  may be mapped from the collective spin  $J$  in a low excitation limit. Note that the different mapping ways depend on the phases of the bath because the bosonization of collective spin is essentially a mean field approach based on choice of the order parameter.

The solution  $|\Psi(t)\rangle$  of the Schrodinger equation  $i\hbar\partial_t |\Psi(t)\rangle = H_{AJC} |\Psi(t)\rangle$  can be expressed as

$$|\Psi(t)\rangle = \sum_{n=0} [c_{\uparrow, n+1}(t) |\uparrow\rangle \otimes |n+1\rangle + c_{\downarrow, n}(t) |\downarrow\rangle \otimes |n\rangle], \quad (25)$$

where  $n = \langle d^\dagger d \rangle$  is the mean “photon” number. A straightforward calculation determines the probability amplitudes [11]

$$c_{\uparrow,n+1}(t) = \{c_{\uparrow,n+1}(0) \left[ \cos(\Omega_n t) - i \frac{\Delta}{\Omega_n} \sin(\Omega_n t) \right] + i \frac{k}{\Omega_n} \sqrt{n+1} c_{\downarrow,n}(0) \sin(\Omega_n t)\} \exp\left(i \frac{\Delta}{2} t\right), \quad (26)$$

$$c_{\downarrow,n}(t) = \{c_{\downarrow,n}(0) \left[ \cos(\Omega_n t) + i \frac{\Delta}{\Omega_n} \sin(\Omega_n t) \right] + i \frac{k}{\Omega_n} \sqrt{n+1} c_{\uparrow,n+1}(0) \sin(\Omega_n t)\} \exp\left(-i \frac{\Delta}{2} t\right), \quad (27)$$

where  $\Delta = \nu + \omega$  and  $\Omega_n = \sqrt{(\Delta/2)^2 + k^2(n+1)}$ .

After a routine calculation, we obtain the purity of the central qubit (with an initial state  $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ ) in the symmetry broken phase ( $0 < \lambda < 1$ ) (see Appendix A)

$$P = \frac{1}{4} \left[ 1 + \cos^2(\Omega_0 t) + \left( \frac{\Delta}{2\Omega_0} \right)^2 \sin^2(\Omega_0 t) \right]^2 + \frac{1}{4} \left[ \left( \frac{k}{\Omega_0} \right)^4 \sin^4(\Omega_0 t) + 2 \left( \frac{k}{\Omega_0} \right)^2 \sin^2(\Omega_0 t) \right]. \quad (28)$$

In Case I ( $\lambda' = \lambda/N$ ), the coupling strength  $k = 2\lambda/\sqrt{N}$  is inversely proportional to the square root of the qubit number  $N$ . In the large  $N$  limit,

$$k^2 = \frac{4\lambda^2}{N} \ll 4\lambda^2 = \Delta^2, \quad (29)$$

i.e.,

$$\lim_{N \rightarrow \infty} \left( \frac{k}{\Omega_0} \right)^2 = \frac{4k^2}{\Delta^2} = 0, \quad (30)$$

$$\lim_{N \rightarrow \infty} \left( \frac{\Delta}{2\Omega_0} \right)^2 = \frac{\Delta^2}{\Delta^2} = 1.$$

Hence from Eq. (28), we have  $P = 1$  in large  $N$  limit. This analytical analysis agrees well with Eq. (22) and Fig. 1.

Generally speaking, the quantum coherence (measured by purity) of a quantum open system would be dissipated by its bath. But when the coupling strength between the system and its bath becomes vanishingly small, the system and bath becomes decoupled, and then the system will preserve all its coherence (remains in a pure state or the purity remains to be unity) during the dynamic evolution. In the thermodynamic limit, the coupling strength between the two level atom and the “radiation field” becomes vanishingly small, i.e., the radiation field and the central qubit are decoupled. Thus, the central qubit evolves under the free Hamiltonian  $H_S = -2s_z$ , which preserves quantum coherence of the central qubit.

In Case II ( $\lambda' = \lambda/\sqrt{N}$ ), the coupling strength  $k = 2\lambda$ . Even in the thermodynamic limit, the interaction Hamiltonian does not vanish ( $2\lambda'\sqrt{N} = 2\lambda \neq 0$ ). This is why

the purity of the central qubit does not preserve in Case II even when the bath is in the symmetry broken phase. In this case, the purity (28) can be simplified to

$$\lim_{N \rightarrow \infty} P = \frac{1}{4} \left[ \frac{32}{25} \sin^4(\sqrt{5}\lambda t) - \frac{8}{5} \sin^2(\sqrt{5}\lambda t) + 4 \right], \quad (31)$$

which is independent of  $N$ , as we observed numerically in Sec. IV.

## B. Symmetric phase

In the above discussion, the system and the bath is reduced into the anti J-C model with a normal coupling, i.e., the coupling does not depend on the mean “photon” number. We now show that the system and the bath can be reduced into the anti J-C model with an intensity-dependent coupling when the bath is in the symmetric phase. In the symmetric phase ( $\lambda > 1$ ), however, the ground state  $|N/2, I(\lambda)\rangle$  of the bath is no longer a low excitation state after the H-P transformation. The mean “photon” number

$$n = \langle d^\dagger d \rangle = \frac{N}{2} - I(\lambda) \approx \frac{N}{2} \left( 1 - \frac{1}{\lambda} \right). \quad (32)$$

is of the same order of  $N$ . By applying H-P transformation the Hamiltonian (6) can be rewritten as

$$H = \frac{2\lambda}{N} (d^\dagger d)^2 + 2(1 - \lambda) d^\dagger d - N - 2\lambda' (s_+ d^\dagger \sqrt{N - d^\dagger d} + s_- \sqrt{N - d^\dagger d} d) - 2s_z. \quad (33)$$

The model described by this Hamiltonian (33) is an intensity-dependent coupling anti J-C model with a Kerr-effect term  $2\lambda(d^\dagger d)^2/N$ , which can be analytically diagonalized as well. After a similar derivation to that in the symmetry broken phase, we obtain the solution  $|\Phi(t)\rangle$  of the Schrodinger equation  $i\hbar\partial_t |\Phi(t)\rangle = H |\Phi(t)\rangle$ ,

$$|\Phi(t)\rangle = \sum_{n=0} [c'_{\uparrow,n+1}(t) |\uparrow\rangle \otimes |n+1\rangle + c'_{\downarrow,n}(t) |\downarrow\rangle \otimes |n\rangle], \quad (34)$$

where

$$\begin{aligned} & c'_{\uparrow,n+1}(t) \\ &= \{c'_{\uparrow,n+1}(0) \left[ \cos(\Omega'_n t) - i \frac{\Lambda'_n}{\Omega'_n} \sin(\Omega'_n t) \right] + ic'_{\downarrow,n}(0) \frac{\Gamma'_n}{\Omega'_n} \sin(\Omega'_n t)\} \exp\{-iA_n t\} \\ & c'_{\downarrow,n}(t) \\ &= \{c'_{\downarrow,n}(0) \left[ \cos(\Omega'_n t) + i \frac{\Lambda'_n}{\Omega'_n} \sin(\Omega'_n t) \right] + ic'_{\uparrow,n+1}(0) \frac{\Gamma'_n}{\Omega'_n} \sin(\Omega'_n t)\} \exp\{-iB_n t\}, \end{aligned} \quad (35)$$

and

$$\begin{aligned}
\Delta' &= 2(1 - \lambda) + (-2) = -2\lambda, \\
\Lambda'_n &= \frac{\lambda}{N}(2n + 1) + \frac{\Delta'}{2}, \\
\Gamma'_n &= 2\lambda'\sqrt{(N - n)(n + 1)}, \\
\Omega'_n &= \sqrt{(\Lambda')^2 + (\Gamma')^2}, \\
A_n &= \frac{\lambda}{N}[n^2 + (n + 1)^2] - \frac{\Delta'}{2}, \\
B_n &= \frac{\lambda}{N}[n^2 + (n + 1)^2] + \frac{\Delta'}{2}.
\end{aligned} \tag{36}$$

The purity of the central qubit (with an initial state  $(|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ ) in the symmetric phase ( $\lambda > 1$ ) can be determined as (see Appendix B)

$$P = \frac{1}{2} + \frac{1}{2} \left[ 1 - \left( \frac{2\lambda'}{\Omega'_n} \right)^2 (N - n)(n + 1) \sin^2(\Omega'_n t) \right]^2. \tag{37}$$

In Case I ( $\lambda' = \lambda/N$ ), the purity  $P$  (37) can be further simplified as

$$P = \frac{1}{2} + \frac{1}{2} \left[ 1 - \left( 1 - \frac{1}{\lambda^2} \right) \sin^2(\lambda t) \right]^2. \tag{38}$$

We see from Eq. (38) that  $P$  varies periodically and is independent of  $N$  though the coupling strength  $-2\lambda'\sqrt{N - d^\dagger d}$  in Eq. (34), as we have seen in Fig. 1. The physics behind Eq. (38) is that the mean “photon” number of the ground state is also  $N$ -dependent, which counterbalances with the  $N$ -dependent coupling strength, leading to the  $N$ -independent dynamical behavior of the purity  $P$  in Case I.

On the other hand, in Case II ( $\lambda' = \lambda/\sqrt{N}$ ), the purity  $P$  (37) can be further simplified as

$$P = \frac{1}{2} + \frac{1}{2} \left\{ 1 - \sin^2 \left[ \sqrt{N(\lambda^2 - 1)} t \right] \right\}^2. \tag{39}$$

Hence, the behavior of purity would not reach a steady pattern when  $N$  increases, as observed in Figs. 3 and 4. The  $N$ -dependence of the purity  $P$  in Case II stems from the coupling strength  $-2\lambda'\sqrt{N - d^\dagger d}$  and the  $N$ -dependent mean “photon” number of the ground state; the  $N$ -dependence of them cannot counterbalance with each other.

## VI. SUMMARY

We have studied the dynamic property of a central qubit coupled to an isotropic Lipkin-Meshkov-Glick bath. Two different types of coupling strength between the central qubit and the bath are considered. In both cases, the QPT of the bath is well revealed by the dynamic behavior of the central qubit. We have found that our model is equivalent to an anti J-C model under H-P transformation when the bath is in the symmetry broken phase.

Especially, when the coupling strength between the central qubit and the bath is inversely proportional to the spin number of the bath, the central spin and the bath becomes decoupled, and the central qubit preserves its quantum coherence all the time. The present study not only demonstrates how the QPT influence the quantum coherence of the central qubit, but also establishes the connection between the LMG model and anti J-C model. In addition, our investigation may propose a new scenario to preserve quantum coherence of a central qubit in experimental implementation of quantum computation.

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## APPENDIX A: DYNAMICS OF PURITY IN THE SYMMETRY BROKEN PHASE ( $0 < \lambda < 1$ )

For the anti J-C Hamiltonian (24), the time evolution of a initial state  $|\Psi(0)\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} \otimes |0\rangle$  can be expressed as

$$\begin{aligned}
|\Psi(t)\rangle &= \frac{ik}{\sqrt{2}\Omega_0} \sin(\Omega_0 t) \exp\left(i\frac{\Delta}{2}t\right) |\uparrow\rangle \otimes |1\rangle \\
&+ \left[ \cos(\Omega_0 t) + \frac{i\Delta}{2\Omega_0} \sin(\Omega_0 t) \right] \\
&\times \exp\left(-i\frac{\Delta}{2}t\right) |\downarrow\rangle \otimes |0\rangle + \frac{|\uparrow\rangle}{\sqrt{2}} \otimes |0\rangle.
\end{aligned} \tag{A1}$$

The reduced density matrix  $\rho^S(t)$  (Eq.(13)) of the system is then found to be

$$\begin{aligned}
\rho^S(t) &= \text{Tr}_B |\Psi(t)\rangle \langle \Psi(t)| \\
&= \frac{1}{2} \left[ 1 + \left( \frac{k}{\Omega_0} \right)^2 \sin^2(\Omega_0 t) \right] |\uparrow\rangle \langle \uparrow| \\
&+ \frac{1}{2} \left[ \cos^2(\Omega_0 t) + \left( \frac{\Delta}{2\Omega_0} \right)^2 \sin^2(\Omega_0 t) \right] |\downarrow\rangle \langle \downarrow| \\
&+ \frac{1}{2} \left[ \cos(\Omega_0 t) - i\frac{\Delta}{2\Omega_0} \sin(\Omega_0 t) \right] \exp\left(i\frac{\Delta}{2}t\right) |\uparrow\rangle \langle \downarrow| \\
&+ h.c.
\end{aligned} \tag{A2}$$

Applying Eq. (14), we obtain the purity  $P$  (Eq.(28)) of the central qubit

$$\begin{aligned}
P &= \frac{1}{4} \left[ 1 + \cos^2(\Omega_0 t) + \left( \frac{\Delta}{2\Omega_0} \right)^2 \sin^2(\Omega_0 t) \right]^2 \\
&+ \frac{1}{4} \left[ \left( \frac{k}{\Omega_0} \right)^4 \sin^4(\Omega_0 t) + 2 \left( \frac{k}{\Omega_0} \right)^2 \sin^2(\Omega_0 t) \right].
\end{aligned} \tag{A3}$$

## APPENDIX B: DYNAMICS OF PURITY IN THE SYMMETRIC PHASE ( $\lambda > 1$ )

For a generalized anti J-C Hamiltonian (Eq.(33)), the time evolution of an initial state  $|\Phi(0)\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} \otimes |n\rangle$  can be expressed as

$$\begin{aligned} |\Phi(t)\rangle = & \frac{1}{\sqrt{2}} \left\{ \cos(\Omega'_{n-1}t) - i \frac{\Xi_{n-1}}{\Omega'_{n-1}} \sin(\Omega'_{n-1}t) \right\} \\ & \exp(-iA_{n-1}t) |\uparrow\rangle \otimes |n\rangle + \frac{i}{\sqrt{2}} \frac{\Gamma'_n}{\Omega'_{n-1}} \sin(\Omega'_{n-1}t) \\ & \exp(-iB_{n-1}t) |\downarrow\rangle \otimes |n-1\rangle \\ & + \frac{1}{\sqrt{2}} \left\{ \cos(\Omega'_{n-1}t) + i \frac{\Xi_n}{\Omega'_n} \sin(\Omega'_{n-1}t) \right\} \\ & \exp(-iB_nt) |\downarrow\rangle \otimes |n\rangle + \frac{i}{\sqrt{2}} \frac{\Gamma'_n}{\Omega'_n} \sin(\Omega'_{n-1}t) \\ & \exp(-iA_nt) |\uparrow\rangle \otimes |n+1\rangle, \end{aligned} \quad (\text{B1})$$

where

$$\Xi_n = \frac{\lambda}{N}(2n+1) + \frac{\Delta'}{2}.$$

In the large  $N$  limit, the reduced density matrix  $\rho^S(t)$  (Eq.(13)) of the central qubit is derived as

$$\begin{aligned} \rho^S(t) &= \text{Tr}_B |\Phi(t)\rangle \langle \Phi(t)| \\ &= \frac{1}{2} |\uparrow\rangle \langle \uparrow| + \frac{1}{2} |\downarrow\rangle \langle \downarrow| + \frac{1}{2} \exp[-i(A_n - B_n)t] \times \\ &\quad \left\{ \cos(\Omega'_{n-1}t) - \frac{i}{\Omega'_n} \left[ \frac{\lambda}{N}(2n+1) + \frac{\Delta'}{2} \right] \sin(\Omega'_{n-1}t) \right\}^2 \\ &\quad \times |\uparrow\rangle \langle \downarrow| + h.c. \end{aligned} \quad (\text{B2})$$

Using Eq. (14), we obtain the purity  $P$  (Eq.(37)) of the central qubit

$$P = \frac{1}{2} + \frac{1}{2} \left[ 1 - \left( \frac{2\lambda'}{\Omega'_n} \right)^2 (N-n)(n+1) \sin^2 \left( \frac{\Omega'_n}{2} t \right) \right]^2. \quad (\text{B3})$$

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